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Sharp Estimates for Lebesgue Constants on Compact Lie Groups

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Suppose G is a n-dimensional compact connected semisimple Lie group and D_R is the spherical Dirichlet kernel on G. We prove the existence of a positive constant K such that

$$\|D_R\|_1 \geqslant K R^{(n-1)/2}.$$

This complements the known result $\|D_R\|_1 \leqslant H R^{(n-1)/2}$. We also prove that for a polyhedral Dirichlet kernel D_N the above inequalities hold with N^p in place of $R^{(n-1)/2}$ (p is the number of positive roots of G). © 1986 Academic Press, Inc.

Estimates for Lebesgue constants on the n-dimensional torus have been known for years. For example, if

$$D_R(t) = \sum_{|n| < R} e^{i(n,t)}$$

is the spherical Dirichlet kernel on $\mathbb{T}^n$, we have

$$C_1 R^{(n-1)/2} \leqslant \|D_R\|_1 \leqslant C_2 R^{(n-1)/2} \tag{1}$$

(with C_1 and C_2 non-depending on R; see [1, 2]). Similar results hold for dilations of more general bodies (see [3] for an elegant proof).

For the setting of compact Lie groups various attempts have been made during the last several years. In a sequence of papers Dreseler [6, 7, 8] proved that for an n-dimensional compact connected Lie group with rank l ($l > 1$) we have, for the spherical Dirichlet kernel D_R,

$$C_1 R^{(l-1)/2} \leqslant \|D_R\|_1 \leqslant C_2 R^{(n-1)/2}. \tag{2}$$

Other estimates have been obtained by Meaney [10]. The estimate from

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below in (2) was improved in [9], where the following inequality is proved:

$$\|D_R\|_1 \geqslant CR^p \tag{3}$$

(here p denotes the cardinality of the set of the positive roots).

Anyhow, by joining (2) and (3) we obtain an estimate for $\|D_R\|_1$ which is sharp only for the case $G = SU_2$. In this paper we prove that for an n-dimensional compact connected semisimple Lie group the estimate (1) holds for spherical Dirichlet kernels (just as in $\mathbb{T}^n$). This seems to be noteworthy if one considers the basic difference between dealing with abelian Fourier analysis and analysis on compact connected semisimple Lie groups (see, e.g., [9]).

In the second part of the paper we prove that for a compact connected semisimple Lie group and for polyhedral Dirichlet kernels the estimate (3) holds and is sharp. Observe that (3) does not involve logarithms, then in this case we do not have an analog with the abelian case.

Notation. Let G be an n-dimensional compact connected semisimple Lie group with rank l and let g be its Lie algebra. Let $\mathscr{T}$ denote a maximal torus of G with Lie algebra t. If it^* is the real vector space of the pure imaginary valued linear forms on t, let $R \subset it^*$ be the set of the roots of G, let P be a system of positive roots, let $S = \{\alpha_1, ..., \alpha_l\}$ be the associated system of simple root and W the Weyl group, i.e., the group generated by the reflections σ_i in the hyperplanes $\alpha_i(H) = 0$. We have $l = \dim \mathscr{T} = \operatorname{rank} G$ and $n = 2p + l$, where $p = \operatorname{card} P$.

We can define in t a positive definite euclidean inner product $(\cdot, \cdot)$ via the Killing form and we identify $\lambda \in it^*$ with the unique $H_\lambda \in t$ such that $\lambda(H) = i(H_\lambda, H)$ for every $H \in t$. So we denote $\lambda(H) = i(\lambda, H)$ and we put $(\lambda, \mu) = (H_\lambda, H_\mu)$. If ξ is a character of $\mathscr{T}$, there exists a unique $\lambda \in it^*$ such that $\xi \circ \exp = e^\lambda$ and we denote by it_e^* the set of such λ's. Moreover, $it_e^* = \{\lambda \in it^* : \lambda(H) \in 2\pi i\mathbb{Z}$, for every $H \in \operatorname{Ker} \exp\}$ is the set of the weights of the representations of G. If we put $H_j = 2iH_{\alpha_j}/\alpha_j(H_{\alpha_j})$ $(j = 1, ..., l)$, Ker exp is generated by the vectors $2\pi H_j$ and the fundamental weights are defined by the relations $\lambda_j(H_k) = i\delta_{jk}$ $(j, k = 1, ..., l)$. We can identify the set $\Sigma = \{\lambda \in it_e^* : \lambda = \sum_{j=1}^{l} m_j \lambda_j,\ m_j \in \mathbb{N}\}$ of the dominant weights with the set of the equivalence classes of unitary irreducible representations of G. A dominant weight λ is non-singular if $m_j > 0$ for every $j = 1, ..., l$. We observe that the fundamental weights $\{\lambda_j\}$ form a basis of it^*, hence we can identify Σ with the lattice $\mathbb{N}^l$ in $\mathbb{R}^l$. Analogously if we choose in t the basis $\{H_j\}$, we can identify t with $\mathbb{R}^l$ and Ker exp with the lattice $2\pi\mathbb{Z}^l$. Therefore, if $\lambda = \sum_{j=1}^{l} m_j \lambda_j$ and $H = \sum_{j=1}^{l} h_j H_j \in t$, we have

$$\lambda(H) = i \sum_{j=1}^{l} m_j h_j. \tag{4}$$

We denote by χ_λ and d_λ the character and the dimension of the representation corresponding to the dominant weight λ. By Weyl formulae we have

$$\chi_\lambda(\exp H) = (A_\beta(\exp H))^{-1} \sum_{\sigma \in W} \det \sigma e^{\sigma(\lambda+\beta)(H)} \qquad H \in t \tag{5}$$

and

$$d_\lambda = \prod_{\alpha \in P} ((\lambda + \beta, \alpha)/(\beta, \alpha)) \tag{6}$$

where $\beta = \frac{1}{2} \sum_{\alpha \in P} \alpha$ and A_β is the Weyl function

$$A_\beta(\exp H) = \sum_{\sigma \in W} \det \sigma e^{\sigma(\beta)(H)} = (-2i)^p \prod_{\alpha \in P} \sin(i\alpha(H)/2). \tag{7}$$

In the sequel we denote by K a constant which may change from line to line.

Main Results

The core of the proof of Theorem 1 follows an idea contained in [3].

Theorem 1. *Let $D_R = \sum_{|\lambda+\beta| \leqslant R} d_\lambda \chi_\lambda$ be the spherical Dirichlet kernel on a compact simply connected semisimple Lie group G. There exist positive constants C_1 and C_2 such that*

$$C_1 R^{(n-1)/2} \leqslant \|D_R\|_1 \leqslant C_2 R^{(n-1)/2}.$$

Proof. The inequality on the right is proved in [8]. To prove the left one we can suppose $l > 1$. Indeed if $l = 1$ we have $G = SU_2$ and $p = 1 = (n-1)/2$ and the thesis follows by (3).

Since D_R is a central function

$$\begin{aligned} I &= \int_G |D_R(g)| \, dg = (\operatorname{card} W)^{-1} \int_{\mathcal{T}} |A_\beta(h)|^2 \, |D_R(h)| \, dh \\ &= (\operatorname{card} W)^{-1} \int_Q \Bigg| A_\beta(\exp H) \sum_{|\lambda+\beta| \leqslant R} \prod_{\alpha \in P} (\lambda+\beta, \alpha)(\beta, \alpha)^{-1} \\ &\quad \times \sum_{\sigma \in W} \det \sigma e^{\sigma(\lambda+\beta)(H)} \Bigg| \, dH \end{aligned} \tag{8}$$

where Q is a fundamental domain centered at the origin. By (8) we easily deduce (see, e.g., [5, Chap 2] that

$$I = K \int_Q \left| A_\beta(\exp H) \prod_{\alpha \in P} D_\alpha \left(\sum_{|\mu| \leqslant R} e^{i(\mu, H)} \right) \right| dH \tag{9}$$

where D_α denotes the derivation with respect to the tangent vector H_α. Arguing as in [11], let $I_0 = \{\lambda \in it^*: |\lambda(H_j)| \leqslant \frac{1}{2}$, for every $j = 1, ..., l\}$. For any $\mu \in it_e^*$ we set $I_\mu = I_0 + \mu$; obviously the interior of $I_\mu \cap I_\nu$ is empty if $\mu \neq \nu$ and the symmetric difference $B_R \Delta C_R$ between $B_R = \bigcup_{|\mu| \leqslant R} I_\mu$ and C_R (the ball of radius R in it^*) is contained in a subset of it^* ($\cong \mathbb{R}^l$), whose (Lebesgue) measure is less than a constant times R^{l-1}. Now

$$\int_{B_R} e^{i(s,H)}\, ds = \sum_{|\mu| \leqslant R} \int_{I_\mu} e^{i(s,H)}\, ds = \int_{I_0} e^{i(s,H)}\, ds \sum_{|\mu| \leqslant R} e^{i(\mu, H)}.$$

An easy computation allows us to obtain

$$\int_{I_0} e^{i(s,H)}\, ds = K \prod_{j=1}^{l} \lambda_j^{-1}(H) \sin(i\lambda_j(H)/2).$$

Hence

$$S(H) = \left(\int_{I_0} e^{i(s,H)}\, ds \right)^{-1}$$

is a C^∞ function in $\bar{Q}$ ($\cong [-\pi, \pi]^l$).

Fix $\Gamma_\varepsilon = \{H \in t: \varepsilon \leqslant |H| \leqslant 2\varepsilon\}$ for a small positive ε which will be chosen later. We have

$$I \geqslant K \int_{\Gamma_\varepsilon} \left| A_\beta(\exp H) \prod_{\alpha \in P} D_\alpha \left(S(H) \int_{B_r} e^{i(s,H)}\, ds \right) \right| dH.$$

We will denote by b_R, c_R, and d_R the characteristic function of B_R, C_R, and $B_R \Delta C_R$, respectively. Obviously

$$\prod_{\alpha \in P} D_\alpha(S \cdot (b_R)\hat{\ }) = S \prod_{\alpha \in P} D_\alpha (b_R)\hat{\ } + \sum_{P'} \prod_{\alpha \in P'} D_\alpha S \prod_{\alpha \in P \setminus P'} D_\alpha (b_R)\hat{\ }$$

where P' is any nonempty subset of P. Hence

$$\begin{aligned} I &\geqslant K \left(\int_{\Gamma_\varepsilon} \left| A_\beta(\exp H) \prod_{\alpha \in P} D_\alpha (b_R)\hat{\ }(H) \right| dH \right. \\ &\qquad \left. - \int_{\Gamma_\varepsilon} \left| A_\beta(\exp H) \sum_{P'} \prod_{\alpha \in P \setminus P'} D_\alpha (b_R)\hat{\ }(H) \right| dH \right) \\ &= I_1 - I_2. \end{aligned}$$

By (7), Bernstein's theorem on entire functions of exponential type and Plancherel formula we obtain

$$\begin{aligned}
&\int_{\Gamma_\varepsilon} \left| A_\beta(\exp H) \prod_{\alpha \in P - P'} D_\alpha(b_R)^\wedge (H) \right| dH \\
&\leqslant K \left(\int_{\Gamma_\varepsilon} |A_\beta(\exp H)|^2 \, dH \right)^{1/2} \left\| \prod_{\alpha \in P - P'} D_\alpha(b_R)^\wedge \right\|_{L^2(\mathbb{R}^l)} \\
&\leqslant K\varepsilon^p (\text{meas } \Gamma_\varepsilon)^{1/2} R^{p - \text{card } P'} \|(b_R)^\wedge\|_{L^2(\mathbb{R}^l)} \\
&\leqslant K\varepsilon^{n/2} R^{-\text{card } P' + n/2}
\end{aligned}$$

where K does not depend on ε or R and meas denotes the Lebesgue measure. Hence we get

$$I_2 \leqslant K_1 \varepsilon^{n/2} R^{(n-2)/2}. \tag{10}$$

Now

$$\begin{aligned}
I_1 &\geqslant K \left(\int_{\Gamma_\varepsilon} \left| A_\beta(\exp H) \prod_{\alpha \in P} D_\alpha(c_R)^\wedge (H) \right| dH \right. \\
&\qquad \left. - \int_{\Gamma_\varepsilon} \left| A_\beta(\exp H) \prod_{\alpha \in P} D_\alpha(d_R)^\wedge (H) \right| dH \right) \\
&= I_3 - I_4.
\end{aligned}$$

If we apply to I_4 the argument we used for I_2 we get

$$\begin{aligned}
I_4 &\leqslant K\varepsilon^{n/2} R^p \|(d_R)^\wedge\|_{L^2(\mathbb{R}^l)} = K\varepsilon^{n/2} (\text{meas}(B_R \Delta C_R))^{1/2} \\
&\leqslant K_2 \varepsilon^{n/2} R^{(n-1)/2}.
\end{aligned} \tag{11}$$

At this point we recall that, if $\tilde{f}(H) = f(|H|)$ is a C^∞ radial functions on t, then

$$\left(\prod_{\alpha \in P} D_\alpha \tilde{f} \right)(H) = \left(\frac{1}{r} \frac{d}{dr} \right)^p f(|H|) \prod_{\alpha \in P} (\alpha, H) \tag{12}$$

(see [5, Lemme 3]). Since

$$\int_{|s| < 1} e^{i(s, H)} \, ds = |H|^{-l/2} J_{l/2}(|H|),$$

it follows by (12) and by the usual formulae for Bessel functions

$$\prod_{\alpha \in P} D_\alpha \left(\int_{C_R} e^{i(s,H)} \, ds \right) = R^{l+p} \prod_{\alpha \in P} D_\alpha \left(\int_{|s|<1} e^{i(s,H')} \, ds \right) \Bigg|_{H'=RH}$$

$$= R^n (R\,|H|)^{-n/2} J_{n/2}(R|H|) \prod_{\alpha \in P} (\alpha, H)$$

$$= K \prod_{\alpha \in P} (\alpha, H) \, (R^{(n-1)/2} \, |H|^{-(n+1)/2}$$

$$\times \cos(R\,|H| - (n+1)\,\pi/4)$$

$$+ R^n O((R\,|H|)^{-(n+3)/2})).$$

Therefore

$$I_3 \geqslant K(R^{(n-1)/2} \int_{\Gamma_\varepsilon} \left| \prod_{\alpha \in P} (\alpha, H) \, A_\beta(\exp H) \cos(R\,|H| - (n+1)\,\pi/4) \right|$$

$$\times |H|^{-(n+1)/2} \, dH$$

$$- R^n \int_{\Gamma_\varepsilon} \left| \prod_{\alpha \in P} (\alpha, H) \, A_\beta(\exp H) \right| O((R\,|H|)^{-(n+3)/2}) \, dH)$$

$$= I_5 - I_6.$$

If we use (7) and we transform I_6 in polar coordinates

$$I_6 \leqslant K\varepsilon^p R^{(n-3)/2} \int_{\Gamma_\varepsilon} \left| \prod_{\alpha \in P} (\alpha, H) \right| |H|^{-(n+3)/2} \, dH$$

$$= K\varepsilon^p R^{(n-3)/2} \int_S \prod_{\alpha \in P} |\alpha(\theta)| \, d\theta \int_\varepsilon^{2\varepsilon} t^{(l-5)/2} \, dt$$

where S is the surface of the unit sphere in t. Therefore

$$I_6 \leqslant K_3 \varepsilon^{(n-3)/2} R^{(n-3)/2}. \tag{13}$$

Finally, we need to estimate I_5. Let Γ'_ε be the intersection of Γ_ε with an open cone strictly contained in the interior of the fundamental Weyl chamber $F = \{H \in t \colon \alpha_j(H) > 0 \text{ for all } j = 1,..., l\}$. So

$$|A_\beta(\exp H)| \geqslant K\varepsilon^p \tag{14}$$

for every H in Γ'_ε and for a suitable constant K. Then

$$I_5 \geqslant KR^{(n-1)/2} \int_{\Gamma'_\varepsilon} \left| \prod_{\alpha \in P} (\alpha, H) \, A_\beta(\exp H) \cos\left(R\,|H| - (n+1)\frac{\pi}{4} \right) \right|$$

$$\times |H|^{(-n-1)/2} \, dH$$

and we handle I_5 as we did for I_6 but we use (14) in place of (7):

$$I_5 \geqslant K\varepsilon^p R^{(n-1)/2} \int_{\varepsilon}^{2\varepsilon} t^{p-(n+1)/2} \,|\cos(Rt-(n+1)\,\pi/4)|\, t^{l-1}\, dt$$

$$= K\varepsilon^p R^{(n-1-(l-1))/2} \int_{R\varepsilon}^{2R\varepsilon} t^{(l-3)/2} \left|\cos\left(t-(n+1)\frac{\pi}{4}\right)\right| dt.$$

We suppose $R\varepsilon > \pi$. By periodicity we get

$$I_5 \geqslant K\varepsilon^p R^{(n-1-(l-1))/2} \sum_{k=1}^{[R\varepsilon/\pi]} \int_{R\varepsilon+(k-1)\pi}^{R\varepsilon+k\pi} t^{(l-3)/2}\, dt$$

and since $l > 1$,

$$I_5 \geqslant K_4 \varepsilon^{(n-1)/2} R^{(n-1)/2}. \tag{15}$$

By (10), (11), (13), and (15) we get, for large values of R (for example $R > \varepsilon^{-2}$),

$$\begin{aligned} I &\geqslant (K_4\varepsilon^{(n-1)/2} - K_3\varepsilon^{(n+1)/2} - K_2\varepsilon^{n/2} - K_1\varepsilon^{(n+2)/2})\, R^{(n-1)/2} \\ &= M(\varepsilon)\, R^{(n-1)/2}. \end{aligned}$$

Now we choose ε in such a way that $M(\varepsilon) > 0$ and the proof is complete.

We now turn to the polyhedral partial sums. Let ω be a dominant weight. We denote by $P'(\omega)$ the set of all the dominant λ's such that $(\lambda_j, \lambda) \leqslant (\lambda_j, \omega)$ for every $j = 1,..., l$. The polyhedron $P(\omega)$ is the union of the satured hulls of the dominant weights $\lambda \in P'(\omega)$:

$$P(\omega) = \bigcup_{\sigma \in W} \sigma P'(\omega).$$

Let N be an integer. We denote by D_N^ω the polyhedral Dirichlet kernel $D_N^\omega = \sum_{\lambda \in P'(N\omega)} d_\lambda \chi_\lambda$ on G.

THEOREM 2. *Let G be a compact simply connected semisimple Lie group. Let ω be a nonsingular dominant weight. Then there exist two positive constants K_1 and K_2 such that*

$$K_1 N^p \leqslant \|D_N^\omega\|_1 \leqslant K_2 N^p.$$

Proof. For the estimate from below we can proceed as in Proposition 3 of [9] or, more directly, in the following way:

$$\begin{aligned} I = \|D_N^\omega\|_1 &= K \int_Q \left| \sum_{\lambda \in P'(N\omega)} d_\lambda \sum_{\sigma \in W} \det \sigma e^{\sigma(\lambda+\beta)(H)} \sum_{\tau \in W} \det \tau e^{\tau(\beta)(H)} \right| dH \\ &= K \int_Q |T_\omega(H)|\, dH. \end{aligned}$$

The coefficient of the exponential $e^{i(N\omega+2\beta,H)}$ in the trigonometric polynomial T_ω is $d_{N\omega}=\prod_{\alpha\in P}(N\omega+\beta,\alpha)/(\beta,\alpha)\geqslant KN^p$. Therefore,

$$I\geqslant K\int_Q T_\omega(H)\,e^{-i(N\omega+2\beta,H)}\,dH\geqslant K_1N^p.$$

On the other hand, in the polyhedral case (9) turns into

$$I=K\int_Q\left|A_\beta(\exp H)\prod_{\alpha\in P}D_\alpha\left(\sum_{\mu\in P(N\omega+\beta)}e^{i(\mu,H)}\right)\right|dH$$

where $D'_N(H)=\sum_{\mu\in P(N\omega+\beta)}e^{i(\mu,H)}$ is a polyhedral Dirichlet kernel on Q ($\cong\mathbb{T}^l$). We observe that

$$\begin{aligned}I&\leqslant K\left(\int_Q\left|\prod_{\alpha\in P}D_\alpha((A_\beta\circ\exp)\,D'_N)(H)\right|dH\right.\\&\qquad\left.-\sum_{P'}\int_Q\left|\prod_{\alpha\in P'}D_\alpha(A_\beta\circ\exp)(H)\prod_{\alpha\in P\setminus P'}D_\alpha D'_N(H)\right|dH\right)\\&=I_1-I_2\end{aligned}$$

where the sum is over all the nonempty subsets P' of P.

D'_N is a trigonometric polynomial whose degree is at most a constant times N in each variable. In fact $|(\mu,H_j)|\leqslant C\,|\mu|$. Since $|\sigma\mu|=|\mu|$ for every $\sigma\in W$ we can suppose μ dominant. If $\mu=\sum_{j=1}^l r_j\alpha_j$ and $N\omega+\beta=\sum_{j=1}^l s_j\alpha_j$ ($s_j\geqslant r_j$, $j=1,\ldots,l$; r_j and s_j not necessarily integers) we have

$$|\mu|^2-|\mu-(s_j-r_j)\,\alpha_j|^2=2(r_j-s_j)(\mu,\alpha_j)-(\alpha_j,\alpha_j)<0. \tag{16}$$

It follows by (16) that $|(\mu,H_j)|\leqslant C\,|N\omega+\beta|\leqslant KN$.

By the classical l-dimensional Bernstein inequality for trigonometric polynomials

$$\begin{aligned}\int_Q&\left|\prod_{\alpha\in P-P'}D_\alpha D'_N(H)\prod_{\alpha\in P'}D_\alpha A_\beta(\exp H)\right|dH\\&\leqslant KN^{p-\operatorname{card}P'}\,\|D'_N\|_{L^1(\mathbb{T}^l)}.\end{aligned}\tag{17}$$

But D'_N is a polyhedral Dirichlet kernel on the l-dimensional torus. We can think the polyhedron $P(N\omega+\beta)$ as the intersection of q hypercubes of edge cN (q and c do not depend on N) and we have

$$\|D'_N\|_{L^1(\mathbb{T}^l)}\leqslant K(\log N)^q.$$

It follows that

$$I_2=o(N^p)\qquad\text{as}\quad N\to\infty. \tag{18}$$

Now we observe that also $D'_N A_\beta$ is a trigonometric polynomial of degree less than a constant times N in each variable. Hence

$$I_1 \leqslant KN^p \|D'_N A_\beta\|_{L^1(T')}. \tag{19}$$

If we prove that there exists a constant C such that

$$\|D'_N A_\beta\|_{L^1(T')} \leqslant C \tag{20}$$

our theorem is a consequence of (18) and (19). The inequality (20) is proved in the following lemma.

LEMMA. *Let* $r = \operatorname{card}\{\mu \in it_c^*\colon\ 0 \leqslant (\lambda_j, \mu) \leqslant 2(\lambda_j, \beta)$ *for all* $j = 1,..., l\}$. *Then the trigonometric polynomial* $D'_N A_\beta$ *is a sum of at most r exponentials (whose coefficients have modulus smaller than* card W).

Proof. We have

$$D'_N(H)\, A_\beta(\exp H) = \sum_{\mu \in P(N\omega + 2\beta)} \left(\sum{}' \det \sigma\right) e^{i(\mu, H)} \tag{21}$$

where $\sum'$ is the sum over all the elements $\sigma \in W$ such that $\mu - \sigma(\beta) \in P(N\omega + \beta)$. We claim that the coefficient of $e^{i(\mu, H)}$ is 0, if μ is not too close to the vertices of the polyhedron (where the expression "too close" does not depend on N).

In view of the symmetry of $P(N\omega + 2\beta)$ it is enough to prove this claim when μ is dominant. We remark that if μ is dominant and $v = \mu - \sigma(\beta)$ for a suitable $\sigma \in W$ then, for large N, $v \in P(N\omega + \beta)$ if and only if $(\lambda_j, v) \leqslant (\lambda_j, N\omega + \beta)$ for every $j = 1,..., l$ (we recall that ω is nonsingular).

We suppose that $\mu \in P'(N\omega + 2\beta)$ satisfies the following condition of "noncloseness" to the vertex $N\omega + 2\beta$:

(*) there exists at least one index $j \in \{1,..., l\}$ such that $(\lambda_j, \mu) \leqslant (\lambda_j, N\omega)$. Now if v is a weight such that $v + \sigma(\beta) = \mu$ for a suitable $\sigma \in W$, then by (*) we have

$$(\lambda_j, v) = (\lambda_j, \mu - \sigma(\beta)) \leqslant (\lambda_j, \mu + \beta) \leqslant (\lambda_j, N\omega + \beta). \tag{22}$$

Moreover,

$$v + 2(\sigma(\beta), \alpha_j)(\alpha_j, \alpha_j)^{-1}\, \alpha_j + \sigma_j \sigma(\beta) = \mu$$

and

$$(\lambda_k, v + 2(\sigma(\beta), \alpha_j)(\alpha_j, \alpha_j)^{-1}\, \alpha_j) = (\lambda_k, v) \qquad \text{if} \quad k \neq j. \tag{23}$$

Obviously, $v \in P(N\omega + \beta)$ or $v \notin P(N\omega + \beta)$. In the first case it follows by (22) and (23) that also $v + 2(\sigma(\beta), \alpha_j)(\alpha_j, \alpha_j)^{-1}\, \alpha_j \in P(N\omega + \beta)$. Moreover, $\det \sigma_j \sigma = -\det \sigma$, hence the corresponding terms in $\sum'$ cancel each other.

On the other hand if $\nu \notin P(N\omega+\beta)$ then $(\lambda_k, \nu) > (\lambda_k, N\omega+\beta)$ for at least an index $k \neq j$. Hence by (23) also $\nu + 2(\sigma(\beta), \alpha_j)(\alpha_j, \alpha_j)^{-1}\alpha_j$ does not belong to $P(N\omega+\beta)$.

As a consequence the coefficient of $e^{i(\mu,H)}$ is zero unless $(\lambda_j, N\omega) \leqslant (\lambda_j, \mu) \leqslant (\lambda_j, N\omega+2\beta)$ for every $j=1,\ldots,l$.

Remark. One could consider polyhedral Dirichlet kernels of the form

$$D_R^{\omega} = \sum_{\lambda \in RP'(\omega)} d_\lambda \chi_\lambda$$

where R is a positive real number. Theorem 2 holds for these kernels, too (we always suppose ω nonsingular). In this case it may exist more than one weight, in $RP(\omega)$, which is maximal with respect to the partial order $\omega_1 \leqslant \omega_2$ ($\omega_1 \leqslant \omega_2$ if $(\lambda_j, \omega_1) \leqslant (\lambda_j, \omega_2)$ for every $j=1,\ldots,l$; we recall that this partial order does not coincide with the usual one: $\omega_1 \preccurlyeq \omega_2$ if $\omega_2-\omega_1$ is a sum of positive roots). The number of maximal (with respect to $\leqslant$) weights in $RP'(\omega)$ does not exceed card$\{\lambda \in it_e^*\colon 0 \leqslant 2(\lambda, \lambda_j)/(\alpha_j, \alpha_j) \leqslant 1,\ j=1,\ldots,l\}$. Moreover if $\{\omega_k'\}$ are the maximal weights in $RP'(\omega)$ it is clear that $d_{\omega_k'} \leqslant KR^p$ for a suitable K (see Proposition 3 in [9] for a similar proof) and that they are not comparable (with respect to $\preccurlyeq$). Then we can split $RP(\omega)$ as disjoint union of the sets $P^*(\omega_k') = \{\lambda \in RP(\omega)\colon \lambda \preccurlyeq \omega_k'\}$ and the proof of Theorem 2 works in this case, too.

We also remark that Theorems 1 and 2 hold for general compact connected semisimple Lie groups (compare with Proposition 3 in [9] and Step 1 of Theorem 2 in [6]).

An Application. The sharp estimates of the previous theorems allow us to extend Theorem B of [4] to the case of compact connected semisimple Lie groups. We refer to [4] for the unexplained notation.

Let E be the space $C(G)$ or the space $L^1(G)$. Suppose f belongs to E and define

$$Xf(g) = \lim_{t \to 0} \frac{f(\exp(-tX)g) - f(g)}{t}$$

in the sense of the norm of E. We denote by $E^{(s)}$ the space of the functions $f \in E$ such that $\prod_{j=1}^{s} X_j f$ exists for any $X_1,\ldots,X_s$ in g. Moreover if $f \in E$, $h \in G$, $m \in \mathbb{N}$, we put

$$\Delta_h^m f(g) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(h^{-j}g)$$

and, for any $t \in \mathbb{R}^+$, we put

$$\omega_m(t, f) = \sup_{d(e,h) \leqslant t} \|\Delta_h^m f\|$$

where d denotes the geodesic distance on G and $\|\cdot\|$ is the norm of E. Finally if $Y_1, \ldots, Y_n$ is a basis for g we define

$$\omega_k(t, f^{(s)}) = \sum_{i_1, \ldots, i_s = 1}^{n} \omega_k(t, Y_{i_1} Y_{i_2} \cdots Y_{i_s} f).$$

We can define spherical and polyhedral partial sums of Fourier series on G as $D_R * f$ and $D_R^{\omega} * f$, respectively.

PROPOSITION. *Let G be a compact connected semisimple Lie group and let $h = (n-1)/2$ (or $h = p$, respectively).*

(a) *If $f \in E^{(s)}$ and $\omega_2(t, f^{(s)}) = o(t^{h-s})$ as $t \to 0$, then the spherical (polyhedral) partial sums of the Fourier series of f converge to f in E.*

(b) *There exists a central function $g \in E^{(s)}$, where s is the largest integer smaller than h, such that $\omega_2(t, f^{(s)}) = O(t^{h-s})$ as $t \to 0$ but such that the spherical (polyhedral) partial sums of the Fourier series of g do not converge to g in E.*

Proof. See [4, Theorem B] and apply Theorem 1 (Theorem 2).

REFERENCES

1. S. A. ALIMOV AND V. A. IL'IN, Conditions for the convergence of spectral decompositions that correspond to self adjoint extension of elliptic operators I, II, *Differential Equations* **7** (1971), 516–543, 651–667.
2. K. I. BABENKO, On the mean convergence of multiple Fourier series and the asymptotic behaviour of the Dirichlet kernel, preprint No. 52, Inst. Prikl. Mat. Akad. Nauk SSSR, Moscow, 1971.
3. M. CARENINI AND P. M. SOARDI, Sharp estimates for Lebesgue constants, *Proc. Amer. Math. Soc.* **89**, (1983), 449–452.
4. D. I. CARTWRIGHT AND P. M. SOARDI, Best conditions for the norm convergence of Fourier series, *J. Approx. Theory* **38** (1983), 344–353.
5. J. L. CLERC, Sommes de Riesz et multiplicateurs sur un groupe de Lie compact, *Ann. Inst. Fourier (Grenoble)* **24** (1974), 149–172.
6. B. DRESELER, Lebesgue constants for spherical partial sums of Fourier series on compact groups *in* "Proceedings, Colloq. on Fourier Anal. Approx. Theory," Budapest, 1976, pp. 327–342.
7. B. DRESELER, Lebesgue constants for certain partial sums of Fourier series on compact Lie groups, *in* "Linear Spaces and Approximation," (P. L. Butzer and B. Sz.-Nagy, Eds.), Birkhäuser, Basel/Stuttgart, 1978.
8. B. DRESELER, Estimates from below for Lebesgue constants of Fourier series on compact Lie groups, *Manuscripta Math.* **31** (1980), 17–23.
9. S. GIULINI, P. M. SOARDI, AND G. TRAVAGLINI, Norms of characters and Fourier series on compact Lie groups. *J. Funct. Anal.* **46** (1982), 88–101.
10. C. MEANEY, Unbounded Lebesgue constants on compact groups, *Monatsh. Math.* **91** (1981), 119–129.
11. V. A. YUDIN, Behavior of Lebesgue constants, *Math. Zametki* **17** (1975), 401–405.

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